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Selberg's Lemma and Applications

1. ABSTRACT

Selberg's Lemma states that given a finitely generated subgroup G of the linear group over $\mathbb{C}$, $GL(n, \mathbb{C})$, i.e., the group of invertible $n \times n$ matrices with entries in the complex numbers, there exists a normal subgroup of G that is torsion-free and of finite index. The beauty of this theorem is that, in its proof, it traverses through group theory, field extensions, and algebraic geometry (a course on commutative algebra), and its applications extend beyond the boundaries of algebra, touching, for example, on geometry and topology. One example is that discrete groups of isometries of hyperbolic space $\mathbb{H}^n$ immediately possess this property (having a subgroup of finite index and torsion-free), which makes the quotient of $\mathbb{H}^n$ by the group's action very well-behaved.

In this minicourse, we will present and prove Selberg's Lemma, covering the necessary prerequisites from ring theory, algebraic geometry, and field extensions, dedicating a good part of the presentation to applications in geometry.

2. MINICOURSE PLAN

Day 1: Historical notes and preliminaries: - Hilbert's Nullstellensatz.

Day 2: More preliminaries: - Field extensions.

Day 3: Selberg's Lemma. - Proof of Selberg's Lemma. - Corollary: Burnside's Theorem (finitely generated and torsion subgroups of the linear group are finite). - What happens with fields of finite characteristic? - The example of $SL(n, \mathbb{Z})$. We will promote Bass-Serre Theory and how we can extract more information from these finite index subgroups of $SL(n, \mathbb{Z})$. - Examples of infinite, finitely generated, and torsion groups (the Gupta-Sidki group).

Day 4: Why do we like torsion-free groups? - Fundamental group and covering spaces. Torsion-free groups tend to have good actions, whose quotients are covering applications.

Day 5: Application - Hyperbolic geometry, models, group of isometries, Poincaré's polygon theorem, geometric interpretation of finite index subgroups. - What happens with negatively curved manifolds that are not constant (Kapovich's example)? - (If time permits) Euclidean geometry and Bieberbach's theorem. It is a theorem of the same nature as Selberg's lemma. It states that given a compact n-manifold of zero sectional curvature, there exists a finite isometric covering to an n-dimensional torus. Algebraically, the fundamental group of the manifold has a finite index subgroup isomorphic to $\mathbb{Z}^n$.

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