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Congruences and cohomology

The course will give an elementary account of p-adic methods in de Rham cohomology of algebraic hypersurfaces with explicit examples and applications in number theory and combinatorics. Lectures are based on a series of our joint papers with Frits Beukers entitled *Dwork crystals* ([1, 2, 3]). These methods also have applications in mathematical physics and arithmetic geometry ([4, 5]).

For an algebraic variety X over a finite field $\mathbb{F}_p$ there exist non-negative integers m, k and algebraic numbers $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_k \in \overline{\mathbb{Q}}$ such that numbers of points on X over all field extensions are given by

$$\#X(\mathbb{F}_{p^s}) = \sum_{i=1}^{m} \alpha_i^s - \sum_{j=1}^{k} \beta_j^s \quad \text{for all} \quad s \geqslant 1.$$

These α_i's and β_j's are called the *Frobenius roots* of X. Their existence in a general setting was proved by Bernard Dwork in early 1960s using p-adic methods. The natural building blocks in Dwork's approach were hypersurfaces. We are going to elaborate explicit p-adic formulas for their Frobenius roots and explore related algebraic constructions.

1. WEIL CONJECTURES AND p-ADIC NUMBERS

We start with explaining why Frobenius roots are interesting and state their properties, which were conjectured in 1940s by André Weil. Weil's conjectures were proved in 1960s and 1970s leading to the creation of p-adic and étale cohomology theories, with contributions by Bernard Dwork, Alexander Grothendieck, Pierre Deligne and many other mathematicians.

In the second part of the lecture we will discuss basic facts about p-adic numbers in preparation for the rest of the course.

2. SOME ELEMENTARY CONGRUENCES

Our main player is a multivariable Laurent polynomial $f(\mathbf{x}) \in R[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ with coefficients in a characteristic zero ring R. For ordinary primes p we will attach to $f(\mathbf{x})$ a natural $h \times h$ matrix Λ_p with coefficients in a p-adic completion of R. Here h is the number of integral internal points in the Newton polytope of $f(\mathbf{x})$. The entries of Λ_p are defined as p-adic limits of certain expressions in the coefficients of powers of $f(\mathbf{x})$. Later (in Lecture 4) we will see that when $R = \mathbb{Z}$ the eigenvalues of Λ_p are the Frobenius roots of the hypersurface $X = \{\mathbf{x} : f(\mathbf{x}) = 0\}$ restricted modulo p. This lecture is based on [6].

3. p-ADIC CARTIER OPERATION ON DIFFERENTIAL FORMS

In the third lecture we will consider differential n-forms on the complement of the hypersurface $f(\mathbf{x}) = 0$ and introduce a p-adic operation on them, the Cartier operation. We will prove that under the *Hasse-Witt condition* the quotients modules of differential n-forms by *formally exact forms* are free modules called *unit-root crystals*.

4. PROOF OF DWORK'S CONGRUENCES

In this lecture we will prove that the matrix Λ_p, which was constructed in an elementary way in Lecture 2, is a matrix of the Cartier operation on the unit-root crystal corresponding to the interior of the Newton polytope. We will see that expansion coefficients of differential forms yield period maps modulo prime powers p^s, which will be our main tool in proving various congruences. As another application, we will prove *Gauss' congruences* for expansion coefficients of rational functions.

5. Beyond the unit root part

So far we developed a method which gives those Frobenius roots which are p-adic units. In the last lecture we will introduce *higher Hasse-Witt conditions* and explain a far going generalization of the previous results. These methods will allow to construct Cartier matrices on the whole de Rham cohomology modules. We will discuss applications and related phenomena of *supercongruences*.

References

[1] F. Beukers, M. Vlasenko, Dwork crystals I, Int. Math. Res. Notices, 2021, 8807–8844

[2] F. Beukers, M. Vlasenko, Dwork crystals II, Int. Math. Res. Notices, 2021, 4427–4444

[3] F. Beukers, M. Vlasenko, Dwork crystals III: from excellent Frobenius lifts towards supercongruences, Int. Math. Res. Notices, 2023, 20433–20483

[4] F. Beukers, M. Vlasenko, On p-integrality of instanton numbers, Pure Appl. Math. Q. 19, no. 1 (2023), 7–44

[5] F. Beukers, M. Vlasenko, Frobenius structure and p-adic zeta function, arXiv:2302.09603

[6] M. Vlasenko, Higher Hasse-Witt matrices, Indag. Math. 29 (2018), 1411–1424