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Group gradings on algebras and modules

Gradings (also known as gradations) have played an important role in algebra for a long time: for example, the grading of the algebra of polynomials by total degree or by multidegree, the grading of a complex semisimple Lie algebra by its root lattice, the natural grading of a crossed product.

Since the 1990's, there has been an increasing interest in classifying gradings by arbitrary groups on various algebras and studying the properties of the resulting graded algebras (for example, their "contractions", graded polynomial identities, graded representations).

This mini-course will start with the general theory of gradings on algebras: we will define the universal group of a grading, refinement and coarsening, isomorphism and equivalence. We will also see some important examples of gradings on associative and Lie algebras.

In the second lecture, we will establish the graded version of Wedderburn Theorem: a G-graded associative ring R is graded-simple and satisfies the descending chain condition on graded left (or right) ideals if and only if R is isomorphic to $\mathrm{End}_D(V)$ where D is a graded-division ring and V is a graded D-module of finite rank. Isomorphisms between such graded rings can be explicitly described, which allows us to reduce their classification to that of graded-division rings with support a subgroup of G. These latter are crossed products of the support with division rings, but classifying them up to isomorphism is difficult in general.

The third lecture will be dedicated to the duality between gradings and actions. In particular, if A is a finite-dimensional algebra (not necessarily associative) over an algebraically closed field of characteristic 0, then a grading on A by a (finitely generated) abelian group G is equivalent to a homomorphism of algebraic groups $\widehat{G} \to \mathrm{Aut}(A)$ where $\widehat{G}$ is the character group of G (a quasitorus). Thus the basic problems of classification of gradings by abelian groups on A can be expressed in terms of the algebraic group $\mathrm{Aut}(A)$ and can be transferred to another algebra B if $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$.

In the fourth lecture, we will classify gradings (up to isomorphism) on finite-dimensional simple associative algebras with involution over an algebraically closed field and use the transfer method mentioned above to classify gradings on classical simple Lie and Jordan algebras.

In the fifth and final lecture, we will survey some classification results for fine gradings (up to equivalence) on simple finite-dimensional Lie and Jordan algebras.