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New foundations for matroid theory and tropical geometry

Prerequisites: Basic knowledge of algebraic geometry.

In a joint program with Matthew Baker (and other, in parts), we develop a new type of algebraic objects and algebraic geometry that captures matroids as vector spaces over the so-called Krasner hyperfield and tropical varieties as schemes over the so-called tropical hyperfield. This theory allows for a reformulation of many concepts and results in a precise analogy to classical linear algebra and algebraic geometry, which simplifies and streamlines previous approaches.

More importantly, this theory leads to new insights in both classical theory and about matroids and tropical geometry. For example: (a) Structural insights into the tropical hyperfield and the sign hyperfield lead to a new proof of Descartes rule of signs and the Newton polygon rule. (b) Structural insights into a new invariant for matroids, which we call its "foundation", have streamlined many known results and produced new results about the representation theory of matroids. (c) The combination of the methods from 1 and 2 has led to a better understanding of realization spaces (i.e. intersections of Grassmannian varieties with linear subspaces). (d) A surprising connection between Lorentzian polynomials (after Branden-Huh) with realization spaces over triangular hyperfields leads to new insights into the topological structure of these spaces. (e) The geometric interpretation of matroid theory in terms of $\mathbb{F}_1$-geometry provides a natural answer to ideas of Jacques Tits who initially proposed the idea of a field with one element.

In this course, we give an introduction into this theory, which encompasses:

LECTURE 1. MOTIVATION AND MATROIDS WITH COEFFICIENTS (AKA BAKER-BOWLER THEORY)

After a motivation for these lectures, we turn to an introduction to matroids, bands, and eventually to matroids with coefficients in bands, enriched with lots of examples.

LECTURE 2. BAND SCHEMES, MODULI OF MATROIDS, TROPICALIZATIONS AS BASE CHANGE

After providing some more in depth discussion of bands, we introduce band schemes as a geometrization of bands. In particular, we explain what the Grassmannian over F1 (the field with one element) is, and why it is the moduli space of matroids. We explain how tropicalization can be viewed as a base change from a non-archimedean field to the tropical hyperfield.

LECTURE 3. REALIZATION SPACES, FOUNDATIONS AND APPLICATIONS

The realization space of a matroid is the locus of the Grassmannian whose non-vanishing coordinates form the bases of the matroid. They are related to Mnev's universality theorem, singularities in moduli spaces (after Vakil), Lafforgue's surgery on Grassmannians, Dressians in tropical geometry and Lorentzian polynomials. The geometry of a realization space is in general highly complicated, but completely controlled by an algebraic invariant of the matroid, which is a particular type of band and which we call the "foundation" of the matroid. In this lecture, we will introduce realization spaces and foundations. As applications, we discuss a short proof of a folklore theorem attributed to Lafforgue. We also show how Descartes' rule of signs and the Newton polygon rule can be derived from structural insights into polynomials over the sign hyperfield and the tropical hyperfield.

LECTURE 4. LORENTZIAN POLYNOMIALS AS MATROIDS OVER TRIANGULAR HYPERFIELDS

Lorentzian polynomials were introduced by Branden and Huh as a mean to simplify and generalize proof techniques for unimodularity of various combinatorial functions that previously relied on deep techniques from algebraic geometry. In this lecture, we introduce Lorentzain polynomials and explain how they relate to matroid representations over triangular hyperfields. In particular, this leads to new structural insights into the space of Lorentzian polynomials. This lecture is based on joint work with Baker, Huh and Kummer.

LECTURE 5. TITS'S DREAM

Algebraic geometry over the so-called "field with one element" was an idea first conceived by Jacquet Tits in the 1950s. In this final lecture, we explain how our approach to F1-geometry, paired with matroid theory, provides a natural framework for the geometric objects that were proposed by Tits, and how they are embedded in the far more mysterious combinatorial flag varieties in the sense of Borovic, Gelfand and White. This lecture is based on joint works with Jarra and with Thas.