Newman's proof of the Prime Number Theorem: a treasure map to a mysterious land

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The Prime Number Theorem (PNT)

The Prime Number Theorem is a milestone in all of mathematics.

Theorem 1.1 (J. S. Hadamard, C.-J. de la Vallée Poussin, 1896)

If $\pi(x)$ denotes the prime-counting function, then $\pi(x) \sim \dfrac{x}{\log(x)}$.

If $\mathrm{li}(x) = \displaystyle\int_2^x \frac{1}{\log(t)}\,dt$ denotes the logarithmic integral function, then

Theorem 1.2 (C.-J. de la Vallée Poussin, 1899)

There is a constant $a > 0$ such that $\pi(x) = \mathrm{li}(x) + O(xe^{-a\sqrt{\log(x)}})$.

J. S. Hadamard and C.-J. de la Vallée Poussin

Jacques Salomon Hadamard

Charles-Jean Étienne Gustave Nicolas
Baron de la Vallée Poussin

Primes in small intervals

It wasn't long before we asked about the problem of the number of primes in small intervals.

Question 1.3

Let $\lambda \in \mathbb{R}$ such that $0 < \lambda < 1$. What can be said about the asymptotic behavior of the function $\pi(x + x^{\lambda}) - \pi(x)$?

The widely accepted conjecture is that

Conjecture 1.4

Let $\lambda \in \mathbb{R}$ such that $0 < \lambda < 1$. Then $\pi(x + x^{\lambda}) - \pi(x) \sim \dfrac{x^{\lambda}}{\log(x)}$.

Some achievements

Hoheisel **[Ho]** was the first to prove that there exists at least one $0 < \lambda < 1$ that makes conjecture 1.4 true. In 1930, he showed that

$$\lambda = \frac{32999}{33000}$$

satisfies conjecture 1.4. To do this, he used a result of J. E. Littlewood **[Li]** on a zero-free region of the Riemann zeta function. The value of λ was subsequently reduced to

$$\lambda = \frac{249}{250}$$

by Heilbronn **[He]** in 1933, and then to

$$\lambda = \frac{5}{8} + \varepsilon,$$

for all $\varepsilon > 0$, by Ingham **[In]** in 1937.

Some achievements

An interesting consequence of Ingham's work is that there is always at least one prime number between two consecutive cubes that are sufficiently large.

Currently, the best result is due to Baker, Harman and Pintz **[BHP]**. In 2001, they showed that for all sufficiently large x, it's true that

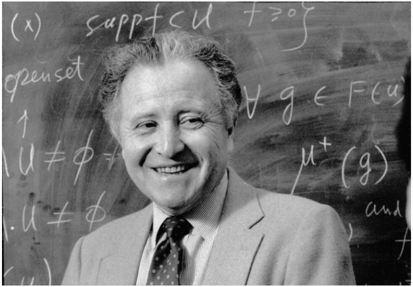
$$\pi(x + x^{0.525}) - \pi(x) \geq \frac{9}{10} \cdot \frac{x^{0.525}}{\log(x)}.$$

The proof of this result is carried out using elements of sieve theory, as well as Buchstab's identity.

Ok... But where does Newman's proof of PNT fit into this story?

Donald Joseph Newman

In 1980, D. J. Newman (**[Ne]**, but also **[Za]**, **[OR]** or **[Su]**) presented us with a short (and very ingenious!) solution to PNT.



Donald Joseph Newman

From now on, I will show that it is possible to use Newman's proof as a treasure map to establish the validity of conjecture 1.4 **for all** $0 < \lambda < 1$!

Step 1: choosing a suitable weight function

Step 1N: Define $\theta(x) = \sum_{p \leq x} \log(p)$. Show that $\theta(x) \sim x$ implies PNT.

Step 1F: Define, for each $c > 0$,

$$W_c(x) = \sum_{p \leq x} \frac{c(1-\lambda)\log(p)\exp(cp^{1-\lambda})}{p^\lambda}.$$

Show that if $W_c(x) \sim \exp(cx^{1-\lambda})$ for some $c > 0$, then

$$\frac{1-\exp[c(\lambda-1)]}{c(1-\lambda)} \leq \liminf_{x\to\infty} \frac{\pi(x+x^\lambda)-\pi(x)}{x^\lambda/\log(x)} \leq$$
$$\leq \limsup_{x\to\infty} \frac{\pi(x+x^\lambda)-\pi(x)}{x^\lambda/\log(x)} \leq \frac{\exp[c(1-\lambda)]-1}{c(1-\lambda)}.$$

In particular, if $W_c(x) \sim \exp(cx^{1-\lambda})$ for some sequence of positive c's converging to 0, then $\pi(x+x^\lambda)-\pi(x) \sim x^\lambda/\log(x)$.

Step 2: the integral lemma

Step 2N: Show that if $\int_1^\infty \frac{\theta(t)-t}{t^2}dt$ converges, then $\theta(x) \sim x$.

Step 2F: Let $0 < c < \frac{2}{1-\lambda}$. Show that if

$$\int_1^\infty \frac{W_c(t) - \exp(ct^{1-\lambda})}{t^\lambda \exp(ct^{1-\lambda})}dt$$

converges, then $W_c(x) \sim \exp(cx^{1-\lambda})$.

Since in the first step we are concerned with the parameter c being close to 0, then the extra condition $c < \frac{2}{1-\lambda}$ is not an obstruction to the work as a whole.

Step 3: change of variables in the integral

Step 3N: Make the change of variables $t = e^x$ and show that

$$\int_1^\infty \frac{\theta(t) - t}{t^2} dt = \int_0^\infty \frac{\theta(e^x)}{e^x} - 1 \ dx.$$

Step 3F: Make the change of variables

$$t = g(x) = [1 + (1 - \lambda)x]^{1/(1-\lambda)}$$

and show that

$$\int_1^\infty \frac{W_c(t) - \exp(ct^{1-\lambda})}{t^\lambda \exp(ct^{1-\lambda})} dt = \int_0^\infty \frac{W_c(g(x))}{\exp[cg(x)^{1-\lambda}]} - 1 \ dx.$$

Obs: Note that $\lim\limits_{\lambda \to 1^-} g(x) = e^x$, so that the change of variables proposed here seems to be a natural generalization of the original.

Step 4: the upper bound lemma

Step 4N: Show that there is a constant $M > 0$ such that $\theta(x) \leq Mx$, for all $x \geq 0$.

Step 4F: Show that for each $c > 0$, there is a constant $K_c > 0$ such that $W_c(x) \leq K_c \cdot \exp(cx^{1-\lambda})$, for all $x \geq 0$.

This step is a consequence of a profound result of Selberg **[Se]**, namely:

Theorem 1.5 (Atle Selberg, 1952)

There is a constant $S > 0$ such that

$$\pi(x+y) - \pi(x) \leq \frac{Sy}{\log(y)}, \qquad \text{for all } x,\ y \geq 2.$$

Obs: It is worth mentioning that Montgomery **[Mo]** showed that we can take $S = 2$ in the previous theorem, although knowing an explicit value for the constant S is not necessary for our purposes.

Step 5: Newman's Analytic Theorem

Theorem 1.6 (D. J. Newman, 1980)

Let $h : [0, \infty) \to \mathbb{R}$ *be a bounded and locally integrable function, whose Laplace transform*

$$\mathscr{L}(s) = \int_0^\infty h(x) e^{-sx} dx,$$

initially defined for all $s \in \mathbb{C}$ *such that* $\Re(s) > 0$*, extends analytically to* $\Re(s) \geq 0$*. Then the improper integral*

$$\int_0^\infty h(x) dx$$

converges and its value is $\mathscr{L}(0)$*.*

Step 5: Newman's Analytic Theorem

Surprisingly, we don't need to change anything in the above theorem!

This may indicate that theorem 1.6 is perhaps a little deeper than it actually appears.

Now, for our next step, let's fix a notation. Define:

- $\Phi(s) = \sum_{p} \frac{\log(p)}{p^s}$, $\forall\ s \in \mathbb{C}$ such that $\Re(s) > 1$.
- $\Psi(s) = \sum_{p} \frac{\log(p)}{p^{\lambda} \exp\left[s\left(\frac{p^{1-\lambda}-1}{1-\lambda}\right)\right]}$, $\forall\ s \in \mathbb{C}$ such that $\Re(s) > 0$.

Step 6: calculating the Laplace transform

Step 6N: The Laplace transform of $h(x) = \dfrac{\theta(e^x)}{e^x} - 1$ is

$$\mathscr{L}(s) = \frac{1}{s+1}\Phi(s+1) - \frac{1}{s},$$

for all $s \in \mathbb{C}$ such that $\Re(s) > 0$.

Step 6F: The Laplace transform of $h(x) = \dfrac{W_c(g(x))}{\exp[cg(x)^{1-\lambda}]} - 1$ is

$$\mathscr{L}(s) = \frac{c(1-\lambda)}{c(1-\lambda)+s}\Psi(s) - \frac{1}{s},$$

for all $s \in \mathbb{C}$ such that $\Re(s) > 0$.

Obs: It is worth noting here that the technique for obtaining this result is the same used by Newman: dividing the interval of integration $[0, \infty)$ into subintervals over which W_c is constant.

Step 7: a clever simplification

Step 7N: The function

$$s \mapsto \frac{1}{s+1}\Phi(s+1) - \frac{1}{s}$$

extends analytically to $\Re(s) \geq 0$ if, and only if, the function

$$s \mapsto \Phi(s) - \frac{1}{s-1}$$

extends analytically to $\Re(s) \geq 1$.

Step 7F: The function

$$s \mapsto \frac{c(1-\lambda)}{c(1-\lambda)+s}\Psi(s) - \frac{1}{s}$$

extends analytically to $\Re(s) \geq 0$ if, and only if, the function

$$s \mapsto \Psi(s) - \frac{1}{s}$$

extends analytically to $\Re(s) \geq 0$.

Step 7: a clever simplification

It is worth noting here the independence of the function $\Psi(s)$ from the parameter c.

Furthermore,

$$\lim_{\lambda\to 1^-}\Psi(s)=\lim_{\lambda\to 1^-}\sum_p \frac{\log(p)}{p^\lambda \exp\left[s\left(\dfrac{p^{1-\lambda}-1}{1-\lambda}\right)\right]}=\sum_p \frac{\log(p)}{p^{s+1}}=\Phi(s+1),$$

for all $s\in\mathbb{C}$ such that $\Re(s)>0$.

This last equality corroborates the fact that we are making a quite natural generalization of Newman's ideas.

The final steps of Newman's proof of PNT

Step 8N: The identity $\displaystyle\sum_p \frac{\log(p)}{p^s - 1} = \Phi(s) + \sum_p \frac{\log(p)}{p^s(p^s - 1)}$ is true for all $s \in \mathbb{C}$ such that $\Re(s) > 1$.

Step 9N: The series $\displaystyle\sum_p \frac{\log(p)}{p^s(p^s - 1)}$ converges absolutely for all $s \in \mathbb{C}$ such that $\Re(s) > 1/2$. In particular, the function

$$s \mapsto \Phi(s) - \frac{1}{s-1}$$

extends analytically to $\Re(s) \geq 1$ if, and only if, the function

$$s \mapsto \sum_p \frac{\log(p)}{p^s - 1} - \frac{1}{s-1}$$

extends analytically to $\Re(s) \geq 1$.

The final steps of Newman's proof of PNT

Step 10N: (Sketch) The function

$$s \mapsto \sum_p \frac{\log(p)}{p^s - 1} - \frac{1}{s-1}$$

in fact extends analytically to $\Re(s) \geq 1$. To do this, we first prove that the identity

$$\sum_p \frac{\log(p)}{p^s - 1} = -\frac{\zeta'(s)}{\zeta(s)}$$

is true for all $s \in \mathbb{C}$ such that $\Re(s) > 1$, and then we show that:

- the function $\zeta(s) - \dfrac{1}{s-1}$ extends analytically to $\Re(s) \geq 1$
- $\zeta(s) \neq 0$, for all $s \in \mathbb{C}$ such that $\Re(s) \geq 1$.

The main obstruction: the analytic continuation

This concludes Newman's proof of the NPT.

But here's the thing! What is the subtle change that would be analogous to step 8N?

If we wanted to continue following very closely in Newman's footsteps, we should now use some kind of algebraic identity (of the type described in step 8N) to finish the proof.

Nonetheless, this would lead us to a problem of commuting an infinite double series (see **[Fe]**) that I honestly don't know how to work around yet. (I'm still working on it!)

Even so, we can complete the proof of conjecture 1.4! How?

Two other simplifications

Step 8F: For $s \in \mathbb{C}$ such that $\Re(s) > 0$, define the functions

$$\Xi(s) = \sum_p \frac{(1-\lambda)\log(p)}{p^\lambda \exp[s(p^{1-\lambda}-1)]} \quad \text{and} \quad \tau(s) = \sum_p \frac{(1-\lambda)\log(p)}{p^\lambda \exp(sp^{1-\lambda})}.$$

Obviously these functions are analytic on $\Re(s) > 0$.

Proposition 1.7 (F.)

a) The function $\Psi(s) - \frac{1}{s}$ extends analytically to $\Re(s) \geq 0$ if, and only if, the function $\Xi(s) - \frac{1}{s}$ extends analytically to $\Re(s) \geq 0$.

b) The function $\Xi(s) - \frac{1}{s}$ extends analytically to $\Re(s) \geq 0$ if, and only if, the function $\tau(s) - \frac{1}{s}$ extends analytically to $\Re(s) \geq 0$.

The final step: the Mellin transform

The Mellin transform is one (of several) integral transformations that exist, and can be seen as a "cousin" of the Laplace transform.

Definition 1.8

We define the Mellin transform of a function $f : (0, \infty) \to \mathbb{R}$ *by*

$$\mathcal{M}_f(z) = \phi(z) = \int_0^\infty x^{z-1} f(x) dx,$$

and the inverse Mellin transform by

$$\mathcal{M}_\phi^{-1}(x) = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-z} \phi(z) dz.$$

The final step: the Mellin transform

Under certain reasonable conditions, we can reobtain the function f from the inverse Mellin transform of its Mellin transform ϕ.

Step 9F: Now, consider the function

$$T(s) = \sum_p \frac{(1-\lambda)\log(p)}{p^{2-\lambda}\exp(sp^{1-\lambda})},$$

obviously analytic in $\Re(s) > 0$.

We will perform two tasks:

- Calculate the Mellin transform of T and the inverse Mellin transform of $\mathcal{M}_T$.
- Check how the previous task relates to our problem.

The final step: the Mellin transform

Proposition 1.9 (F.)

The function

$$\tau(s) - \frac{1}{s} = \sum_p \frac{(1-\lambda)\log(p)}{p^{\lambda}\exp(sp^{1-\lambda})} - \frac{1}{s},$$

initially defined in $\Re(s) > 0$*, extends analytically to* $\Re(s) > -1$*.*

Proof: As mentioned in the previous slide, calculating the Mellin transform of the function $T(s)$ in $\Re(s) > 0$, we get

$$\mathscr{M}(z) = \mathscr{M}\left(\sum_p \frac{(1-\lambda)\log(p)}{p^{2-\lambda}\exp(sp^{1-\lambda})}\right) = \sum_p \frac{(1-\lambda)\log(p)}{p^{2-\lambda}}\mathscr{M}[\exp(-sp^{1-\lambda})]$$

$$= \sum_p \frac{(1-\lambda)\log(p)}{p^{2-\lambda}} \cdot \frac{\Gamma(z)}{p^{(1-\lambda)z}} = (1-\lambda)\Gamma(z)\sum_p \frac{\log(p)}{p^{1+(1-\lambda)(z+1)}}$$

$$= (1-\lambda)\Gamma(z)\Phi(1+(1-\lambda)(z+1)).$$

The final step: the Mellin transform

Note that this function is meromorphic in $\Re(z) > -1$ with a simple pole in $z = 0$, not because of the function

$$\Phi(1 + (1 - \lambda)(z + 1)),$$

which is analytic in $\Re(z) > -1$, but because of the function $\Gamma(z)$ with its simple pole in $z = 0$. Remembering that

$$\Phi(1 + z) = \frac{1}{z} + \Omega(z), \qquad \text{for all } z \in \mathbb{C} \text{ such that } \Re(z) > 0,$$

we can write

$$\begin{aligned} \mathscr{M}(z) &= (1 - \lambda)\Gamma(z)\left[\frac{1}{(1 - \lambda)(z + 1)} + \Omega((1 - \lambda)(z + 1))\right] \\ &= \frac{\Gamma(z)}{z + 1} + (1 - \lambda)\Gamma(z)\Omega((1 - \lambda)(z + 1)). \end{aligned}$$

The final step: the Mellin transform

Now notice that this function has exponential decay over vertical lines, because $\Gamma(z)$ does, and the function $\Omega((1-\lambda)(z+1))$ doesn't change that. Therefore, we can reobtain the function $T(s)$ by calculating the inverse Mellin transform of $\mathscr{M}(z)$, and write

$$\begin{aligned} T(s) &= \mathscr{M}^{-1}(\mathscr{M}(z)) \\ &= \mathscr{M}^{-1}\left(\frac{\Gamma(z)}{z+1} + (1-\lambda)\Gamma(z)\Omega((1-\lambda)(z+1))\right) \\ &= \mathscr{M}^{-1}\left(\frac{\Gamma(z)}{z+1}\right) + (1-\lambda)\mathscr{M}^{-1}[\Gamma(z)\Omega((1-\lambda)(z+1))] \\ &= \exp(-s) - s\cdot\Gamma(0,s) + (1-\lambda)\underbrace{\mathscr{M}^{-1}[\Gamma(z)\Omega((1-\lambda)(z+1))]}_{\text{analytic in } \Re(s)>-1}, \end{aligned}$$

for all $s \in \mathbb{C}$ such that $\Re(s) > 0$. Differentiating this equality twice, we get

The final step: the Mellin transform

$$
\begin{aligned}
\tau(s) &= T''(s) \\
&= \frac{d^2}{ds^2}\left[\exp(-s) - s\cdot\Gamma(0,s)\right] + (1-\lambda)\frac{d^2}{ds^2}\mathscr{M}^{-1}[\Gamma(z)\Omega((1-\lambda)(z+1))] \\
&= \frac{\exp(-s)}{s} + (1-\lambda)\underbrace{\frac{d^2}{ds^2}\mathscr{M}^{-1}[\Gamma(z)\Omega((1-\lambda)(z+1))]}_{\text{analytic in } \Re(s)>-1}.
\end{aligned}
$$

Finally, subtracting $1/s$ on each side of this equation, we get

$$
\tau(s) - \frac{1}{s} = \frac{\exp(-s)-1}{s} + (1-\lambda)\underbrace{\frac{d^2}{ds^2}\mathscr{M}^{-1}[\Gamma(z)\Omega((1-\lambda)(z+1))]}_{\text{analytic in } \Re(s)>-1}.
$$

But since the function $s \mapsto \dfrac{\exp(-s)-1}{s}$ extends analytically to $\mathbb{C}$, we are done. □

The final step: the Mellin transform

Theorem 1.10 (F., 2023)

Let $\lambda \in \mathbb{R}$ *be such that* $0 < \lambda < 1$. *Then*

$$\pi(x + x^{\lambda}) - \pi(x) \sim \frac{x^{\lambda}}{\log(x)}.$$

Proof: [Fe] Since

$$\tau(s) - \frac{1}{s}$$

extends analytically to $\Re(s) > -1$, then

$$\Psi(s) - \frac{1}{s}$$

extends analytically to $\Re(s) \geq 0$, thereby proving the theorem. □

Some cool consequences (for n sufficiently large)

Conjecture 1.11 (Legendre, $\sim$ 1800)

There are always at least two prime numbers between two consecutive squares.

Conjecture 1.12 (Oppermann, 1877)

$\pi(n^2 - n) < \pi(n^2) < \pi(n^2 + n)$, *for all* $n \geq 2$.

Some cool consequences (for n sufficiently large)

Conjecture 1.13 (Sierpiński, 1958)

Let n be an integer greater than 1. If you write the numbers $1, 2, \ldots, n^2$ in a square matrix as follows:

$$\begin{array}{cccc} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ 2n+1 & 2n+2 & \cdots & 3n \\ \cdots & \cdots & \cdots & \cdots \\ (n-1)n+1 & (n-1)n+2 & \cdots & n^2 \end{array}$$

then each line contains at least one prime number.

Conjecture 1.14 (Andrica, 1986)

Denote by p_n the nth prime number. Then $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$, for all $n \geq 1$.

What's next?

The more attentive viewer may have noticed that the proof given by Newman for the PNT is, in fact, an algorithm to solve a problem that can be posed in a very general way: it starts with a *length function*

$$\ell : [2, \infty) \to (0, \infty)$$

for which we want to know, at least asymptotically, how many prime numbers there are in the interval

$$(x, x + \ell(x)].$$

What's next?

The first step of the algorithm is to find a *weight function*

$$w : \mathbb{P} \to (0, \infty)$$

with none, one (or perhaps more than one if necessary) parameter c such that $w = w_c$ and that will assign a weight $w(p)$ for each prime p. This function w will generate a cumulative sum function

$$W(x) = \sum_{p \leq x} w(p),$$

for which, according to **[RS]**, we will try to show that it satisfies the expected approximation

$$W(x) \sim \int_2^x \frac{w(t)}{\log(t)} dt,$$

because showing this relation may imply

$$\pi(x + \ell(x)) - \pi(x) \sim \frac{\ell(x)}{\log(x)}.$$

What's next?

The rest of the algorithm should be easy to understand, but not necessarily easy to execute (especially in the part of the analytical continuation), for those who have followed along this far. As a first test to see if this algorithm is in fact reasonably effective, we invite the enthusiastic viewer to think about the following three problems.

Problem 1.15

Fix $C > 0$*. What happens in the case* $\ell(x) = Cx^{\lambda}$*?*

Problem 1.16

Can the ideas presented here be used to prove what would be the equivalent of the PNT for arithmetic progressions in small intervals?

What's next?

Problem 1.17

*In light of **[Kn]**, is it possible to reproduce what was done here in an abstract way for arithmetic semigroups and arithmetic formations?*

A natural continuation of this work might be to investigate the validity of theorem 1.10 for functions $\ell(x)$ that grow more slowly than x^{λ}, for all $0 < \lambda < 1$, such that

$$\exp[(\log(x))^{\mu}],$$

fore some $0 < \mu < 1$, or even

$$\log(x)^{\log(\log(x))}.$$

What's next?

However, it is worth mentioning that, due to Maier's Theorem **[Ma]**, this algorithm must have some limitation.

Theorem 1.18 (Maier, 1985)

Let $\mu > 1$, *and set* $T(x) = (\log(x))^{\mu}$. *Then*

$$\liminf_{x\to\infty} \frac{\pi(x+T(x)) - \pi(x)}{T(x)/\log(x)} < 1 \quad \textit{and} \quad \limsup_{x\to\infty} \frac{\pi(x+T(x)) - \pi(x)}{T(x)/\log(x)} > 1.$$

Furthermore, if $1 < \mu < e^{\gamma}$, *then*

$$\limsup_{x\to\infty} \frac{\pi(x+T(x)) - \pi(x)}{T(x)/\log(x)} > \frac{e^{\gamma}}{\mu},$$

where $\gamma \approx 0,5772156649...$ *denotes the Euler-Mascheroni constant.*

What's next?

Therefore, we also propose the following problem:

Problem 1.19

What happens in the critical case $\ell(x) = \log(x)^2$?

Finally, we close this talk with the following question to the viewer:

Problem 1.20

How does the function $\tau(s)$ behave in a neighborhood of $\Re(s) = 0$ in the limiting case $\lambda = 0$, that is, how does the function

$$s \mapsto \sum_p \frac{\log(p)}{\exp(sp)}$$

behave near $\Re(s) = 0$? Does this have any significance regarding the distribution of prime numbers?

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Thank you all very much for your attention!

Any questions?

☺

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